

# Symmetry-seeking spacetime coordinates

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In numerically constructing a spacetime that has an approximate timelike Killing vector, it is useful to choose spacetime coordinates adapted to the symmetry, so that the metric and matter variables vary only slowly with time in these coordinates. In particular, this is a crucial issue in numerically calculating a binary black hole inspiral. An approximate *homothetic* vector plays a role in critical gravitational collapse. We summarize old and new suggestions for finding such coordinates from a general point of view. We then test some of these in various toy models with spherical symmetry, including critical fluid collapse and critical scalar field collapse.

## I. INTRODUCTION

One of the fundamental problems in numerically constructing spacetimes without symmetries (3D numerical relativity) is the choice of suitable coordinates. The task is not just to find a good coordinate system for a known spacetime, but the spacetime is not actually known when the calculation begins, so that one has to construct the coordinate system along with the spacetime. In principle, once one has obtained the spacetime in some coordinate system, one can go back over it and paint on a better coordinate system. In practice, the initial attempt must not be too bad, or the coordinate system will break down altogether before one has evolved very far.

One begins the numerical construction of a spacetime with Cauchy data (three-metric  $g_{ab}$ , extrinsic curvature  $K_{ab}$ , and perhaps suitable matter data) obeying the Hamiltonian and momentum constraints, in hand. One then has to make a choice of the lapse  $\alpha$  and shift  $\beta^a$  based on this information in order to start constructing the spacetime along with the coordinate system. For the purpose of this paper, a coordinate condition is a prescription that maps Cauchy data on a Cauchy surface to lapse and shift fields on that Cauchy surface. One then uses the same prescription again on each time slice. We call such coordinate conditions local in time. Typically they are elliptic (in space), and one specifies suitable boundary conditions at the outer (large radius) and inner (black hole excision) boundaries of the numerical domain.

In a generalization of local-in-time coordinate conditions, one may use a prescription for the first or second time derivatives of the lapse and/or shift, obtaining a heat equation or wave equation-like prescription. Such evolution equations for the lapse and shift require much less computational work than solving an elliptic equation at each time step. Coordinate conditions for the purpose of our paper are then elliptic, parabolic or wave equations for the lapse and shift in terms of given Cauchy data on a slice.

What are the characteristics of good coordinate conditions? Here we propose the following desirable criterion: If the spacetime is in fact stationary, the coordinate condition should make the metric coefficients explicitly time-independent, independently of how the initial slice is embedded in the spacetime. If there is an approximate timelike Killing vector, the metric should be approximately time-independent. If there is no Killing vector, the natural extension of our proposal is to minimize some measure of the rate of change of the metric with time.

Consider the inspiral phase of a binary system. There is an approximate timelike Killing vector here, in the sense that all metric coefficients can be made to evolve on the inspiral time scale instead of the much shorter orbital time scale. Clearly, this defines a corotating coordinate system. If the spacetime has an approximate periodicity (discrete isometry connecting timelike related points), as in the case of a highly elliptic inspiral, the metric should become approximately periodic in time through our coordinate condition.

A coordinate system adapted to the presence of a timelike Killing vector, as we have just described it, is far from unique. It is easy to see that the family of such coordinate systems is equivalent to the family of smooth spacelike slices containing one arbitrary but fixed spacetime point, together with spatial coordinates on the slice: the coordinate system on the spacetime is obtained by Lie-dragging the slice and its coordinates along the Killing vector.

One may know that the final state of an evolution (for example a binary black hole system) is a Kerr solution, but this does not mean that the metric coefficients approach the Kerr metric in any of the standard coordinate systems even as they become asymptotically time-independent. We require only that our coordinate condition evolves any given initial slice along the timelike Killing vector, not that it “straightens out” the slice. A “straightest” slice in this

spacetime could be defined as one that minimizes for example  $\int {}^{(3)}R^2 dV$ , but we shall not try to minimize this at the same time.

A mathematically similar problem to that of finding good coordinates in numerical relativity arises in the theory of critical phenomena in gravitational collapse. The entire rich phenomenology of critical collapse can be explained and quantitatively predicted using ideas taken from renormalisation group theory and the theory of dynamical systems [1]. Implicit in this theoretical framework is the concept of a coordinate prescription (local in time) such that the metric is “time”-independent if the Cauchy data generate a spacetime that is homothetic, or continuously self-similar (CSS). We have set “time” in quotation marks here, as  $\partial/\partial t$  is not everywhere timelike in a homothetic spacetime. In this context we are only interested in a prescription that is local in time (in the sense of not containing time derivatives of the lapse and shift). Such a prescription turns general relativity into a dynamical system, with CSS spacetimes corresponding to fixed points. If the spacetime is discretely self-similar (DSS), we require our prescription to make the metric periodic in “time”, with a DSS spacetime corresponding to a limit cycle. (The use of maximal slicing in critical gravitational collapse [2] does part of the job of turning general relativity into the appropriate sort of dynamical system. However, one is still left with the question of what spatial coordinates to use).

Again, such a coordinate system is far from unique. In the neighborhood of any data set generating a CSS or DSS spacetime one can define such a coordinate prescription perturbatively. This perturbative definition has been enough in order to calculate critical exponents to several significant digits. A coordinate condition applicable to all of superspace, with the property that the fixed points of the resulting dynamical system are precisely CSS spacetimes and the limit cycles the DSS spacetimes, would define a renormalisation group for general relativity. Studying its properties would help us in understanding the general dynamics of general relativity and issues relating to cosmic censorship.

In this paper, we study the two related cases in a common framework. In section II, we discuss a number of candidate coordinate conditions in the absence of symmetries (other than stationarity or homothety). Our basic strategy is to write down the condition that the metric is (conformally) time-independent in a 3+1 split, and then choose four equations out of that set of equations to solve for the lapse and shift or their time derivatives. This will guarantee that our criterion is obeyed. If a given prescription obeying this criterion is otherwise a good coordinate condition must be determined empirically. In section III, we specialize the general proposals to spherical symmetry, and we discuss additional proposals that are specific to spherical symmetry. In section IV, we test some of these proposals by evolving a slice in a given spherically symmetric spacetime. (This procedure is simpler and more stable than evolving the Cauchy data themselves.) We use several spacetimes to test the stationary case, the CSS case, and the DSS case.

## II. COORDINATE CHOICES

### A. 3+1 split of the Killing and homothetic equation

We begin with a homothetic Killing vector  $X^a$ . That is

$$\mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a = 2\sigma g_{ab}, \quad (1)$$

where  $\sigma$  is a constant. For  $\sigma = 0$ ,  $X^a$  is a Killing vector. For  $\sigma \neq 0$  a constant we can, without loss of generality, choose  $\sigma = -1$  by rescaling  $X^a$ . In the following  $\sigma$  only takes the values 0 and  $-1$ . We perform the usual 3+1 split of the Einstein equation with  $X^a$  as the evolution vector field. That is, spacetime is foliated by surfaces  $\Sigma(t)$  with  $X^a = (\partial/\partial t)^a$ . Each surface has a unit normal vector  $n^a$ , intrinsic metric  $h_{ab} = g_{ab} + n_a n_b$  with associated derivative operator  $D_a$  and extrinsic curvature  $K_{ab} = -(1/2)\mathcal{L}_n h_{ab}$ . (There is no standard convention for the sign of  $K_{ab}$ . Here we use the sign prevalent in numerical relativity.) We decompose  $X^a$  as  $\alpha n^a + \beta^a$  with  $\beta^a n_a = 0$ . We want to know how the quantities  $(\alpha, \beta^a, h_{ab}, K_{ab})$  evolve with time. We will get one answer from the homothetic Killing equation and another answer from the usual ADM equations. The condition that these two answers agree then gives conditions that these quantities must satisfy in a spacetime with a homothetic Killing vector. [3,4].

We begin by deriving evolution equations from the homothetic Killing equation. From the decomposition of  $X^a$ , it follows that  $n_a = -\alpha \nabla_a t$  and therefore that

$$\mathcal{L}_X n_a = \alpha^{-1}(\mathcal{L}_X \alpha) n_a. \quad (2)$$

Since  $\mathcal{L}_X n_a$  is a scalar times  $n_a$ , it follows from equation (1) that

$$\mathcal{L}_X n_a = \sigma n_a, \quad (3)$$

and therefore that

$$\mathcal{L}_X \alpha = \sigma \alpha. \quad (4)$$

Then using  $0 = \mathcal{L}_X X^a = 0$  we find

$$\mathcal{L}_X \beta^a = 0. \quad (5)$$

It follows from equations (1) and (3) that

$$\mathcal{L}_X h_{ab} = 2\sigma h_{ab}. \quad (6)$$

We then find

$$\mathcal{L}_X K_{ab} = -\frac{1}{2} \mathcal{L}_X \mathcal{L}_n h_{ab} = -\frac{1}{2} (\mathcal{L}_n \mathcal{L}_X h_{ab} + \mathcal{L}_{[X,n]} h_{ab}), \quad (7)$$

and therefore

$$\mathcal{L}_X K_{ab} = \sigma K_{ab}. \quad (8)$$

### B. Solving for the lapse and shift locally in time

We begin by looking at those components of the homothetic Killing equation that contain time derivatives of the lapse and shift. With the shorthand  $\mathcal{L}_X T = \dot{T}$  for any tensor  $T$ , these are

$$\dot{\alpha} = \sigma \alpha, \quad \dot{\beta}^a = 0. \quad (9)$$

Given a good initial guess, we could consider these as evolution equations for the lapse and shift, but this is unlikely to make for a stable scheme when the symmetry is only approximate and/or the initial guess is not perfect. Furthermore, it leaves us with the problem of finding a good initial value for the lapse and shift. Conversely, if we have found a lapse and shift on the initial time slice by solving some elliptic equations, the only reason not to use the same prescription on each time slice would be numerical effort. [If numerical effort is a concern, one could in effect evolve the lapse and shift by (9) for a few time steps, before using the expensive prescription again.]

We now concentrate on the other components of the homothetic Killing equations. The ADM evolution equations are

$$\dot{h}_{ab} \equiv D_a \beta_b + D_b \beta_a - 2\alpha K_{ab}, \quad (10)$$

$$\dot{K}_{ab} \equiv \beta^c D_c K_{ab} + K_{ac} D_b \beta^c + K_{bc} D_a \beta^c - D_a D_b \alpha + \alpha \left( {}^{(3)}R_{ab} + K K_{ab} - 2K_{ac} K_b{}^c - \tau_{ab} - \frac{1}{2}(\rho - \tau) h_{ab} \right), \quad (11)$$

for any vector field  $X^a$ . Here  $\rho \equiv G_{ab} n^a n^b$ ,  $\tau_{ab} \equiv h_a{}^c h_b{}^d G_{cd}$ , and  $J_a \equiv -h_a{}^c n^b G_{cb}$ . Note that in the ADM evolution equations,  $\alpha$  and  $\beta^a$  can be freely specified, so no new information is obtained by combining the ADM equations with the equations (9). But if we substitute the ADM equations into the components

$$\dot{h}_{ab} = 2\sigma h_{ab} \quad (12)$$

$$\dot{K}_{ab} = \sigma K_{ab}. \quad (13)$$

of the homothetic Killing equation, we obtain two symmetric tensor equations which link the lapse and shift, and their spatial derivatives, to the Cauchy data, and their spatial derivatives, all within a single Cauchy surface.

Our task can now be formulated as follows. Out of these 6+6 equations we want to obtain 3+1 equations that can be solved for the lapse and shift, for given generic Cauchy data. The remaining 8 equations remain as consistency conditions which are obeyed if and only if the Cauchy data actually evolve into a spacetime with a (conformal) Killing vector. In other words, given Cauchy data on a slice, we want to have a prescription to calculate a lapse and shift such that one obtains a reasonable coordinate system on the spacetime evolved from “reasonable” generic data, and coordinates adapted to the symmetry if there is a timelike Killing vector or homothetic vector.

Clearly, there is an infinite range of possibilities. In a situation without symmetries, our task is to make one 3-vector (the equation for the shift) and one scalar (the equation for the lapse) out of two symmetric 3-tensors. It is important to note that the equations for the lapse and shift will generally be coupled, so that the same equation for, say, the shift can have a different character when coupled to a different equation for the lapse. Some or all of the resulting equations can be purely algebraic. Alternatively, they can involve first or higher spatial derivatives of the lapse and shift. An elliptic equation or equations would be particularly appealing. Any prescription obtained for the general case can be reduced to spherical symmetry, and we shall do this for testing. Additional prescriptions that explicitly use the spherical symmetry do not generalize back to the case without symmetries.

One can obtain a 3-vector of equations from a 3-tensor by taking a divergence, for example

$$D^a (\dot{h}_{ab} - 2\sigma h_{ab}) = D^a \dot{h}_{ab} = 0. \quad (14)$$

This equation can be derived from varying the action

$$\int d^3x \sqrt{h} h^{ac} h^{bd} (\dot{h}_{ab} - 2\sigma h_{ab}) (\dot{h}_{cd} - 2\sigma h_{cd}) \quad (15)$$

with respect to the shift. As  $\sigma$  drops out of (14), it could also have been derived from varying the action

$$\int d^3x \sqrt{h} h^{ac} h^{bd} \dot{h}_{ab} \dot{h}_{cd}. \quad (16)$$

Condition (14) was first suggested as a shift condition by Smarr and York [5] under the name minimal strain shift. As it is derived from a variational principle, one may hope that it gives good spatial coordinates also in the absence of an exact symmetry. Written out explicitly, minimal strain shift is

$$D_b D^b \beta^a + D_b D^a \beta^b - 2D_b (\alpha K^{ab}) = 0. \quad (17)$$

Taking the divergence of (13), we obtain the 3-vector equation

$$D^a (\dot{K}_{ab} - \sigma K_{ab}) = 0. \quad (18)$$

At first sight this is less appealing than (14) because it contains third spatial derivatives of the lapse.

In order to obtain a scalar equation for the lapse, we can take the trace of (12);

$$h^{ab} (\dot{h}_{ab} - 2\sigma h_{ab}) = 0 \quad (19)$$

is equivalent to

$$\dot{\sqrt{h}} - 3\sigma \sqrt{h} = 0, \quad (20)$$

that is, it determines the scaling of the 3-volume element. ( $h$  is the determinant of  $h_{ab}$ .) Written out in terms of  $\alpha$  and  $\beta^a$ , this equation is

$$D_a \beta^a - \alpha K - 3\sigma = 0. \quad (21)$$

This does not seem to be in general a good equation for the lapse (since it is ill defined where  $K$  vanishes), nor (by virtue of being a scalar) for the shift.

The equation resulting from instead contracting (12) with  $K_{ab}$ ,

$$K^{ab} (\dot{h}_{ab} - 2\sigma h_{ab}) = 0 \quad (22)$$

can be derived from varying the functional (15) with respect to the lapse. We shall refer to Eq. (22) on its own as the conformal strain lapse for  $\sigma = -1$ , or the minimal strain lapse for  $\sigma = 0$ . It is an algebraic equation for the lapse,

$$\alpha = \frac{K^{ab} D_a \beta_b - \sigma K}{K^{ab} K_{ab}}, \quad (23)$$

which is linear in the shift. The fact that the lapse may be zero, negative, or even blow up (where  $K_{ab} = 0$ ), is worrisome. Clearly this prescription cannot be used on all initial data. For the Killing case,  $\sigma = 0$ , Brady, Creighton

and Thorne (BCT) [6] have recently proposed using the minimal strain lapse and minimal strain shift together, as both are derived from the same variational principle (16). If one substitutes the formal minimal strain lapse (23) into the minimal strain shift equation (17), it is not clear if the resulting differential equation for the shift alone is elliptic. We shall refer to the combination of minimal strain lapse and minimal strain shift, in the Killing case  $\sigma = 0$ , as the BCT gauge, and for any  $\sigma$  as the generalized BCT gauge.

The equation

$$h^{ab} (\dot{K}_{ab} - \sigma K_{ab}) = 0 \quad (24)$$

is an elliptic equation for the lapse, and seems not to have been considered before. We shall call it the trace-Kdot lapse. Adding to this the minimal strain lapse equation, however, we obtain

$$0 = h^{ab} (\dot{K}_{ab} - \sigma K_{ab}) - K^{ab} (\dot{h}_{ab} - 2\sigma h_{ab}) = \dot{K} + \sigma K. \quad (25)$$

For  $\sigma = 0$ , this is known as constant (extrinsic) curvature slicing. If  $K = 0$  initially, and therefore at all times, we have maximal slicing. For  $\sigma = -1$  we could speak of exponential curvature slicing. Written out in terms of the Cauchy data, the equation is

$$-D_a D^a \alpha + \left[ {}^{(3)}R + K^2 + \frac{1}{2}(\tau - 3\rho) \right] \alpha + \beta^a D_a K + \sigma K = 0. \quad (26)$$

Note that we do not assume that  $K$  is constant within each slice. We shall refer to the combination of equations (17) and (26) as the generalized Smarr-York (SY) gauge. For an initial  $K = 0$  slice, the generalized SY gauge reduces to maximal slicing with minimal strain.

The differential equations for lapse and shift must be supplemented with boundary conditions in order to yield a solution. For an approximate timelike Killing field in an asymptotically flat spacetime, the condition  $\alpha \rightarrow 1$  and  $\beta^a \rightarrow 0$  at infinity should be reasonable. For the case of binary black hole inspiral, BCT [6] have proposed a boundary condition to use with their equation. It is less clear what would be a reasonable boundary condition to use for the case of critical gravitational collapse.

### III. SPHERICAL SYMMETRY

#### A. Reduction to spherical symmetry

In spherical symmetry, the metric has the form

$$ds^2 = (-\alpha^2 + \beta^r \beta_r) dt^2 + 2\beta_r dr dt + g_{rr} dr^2 + g_{\theta\theta} (d\theta^2 + \sin^2 \theta d\phi^2) \quad (27)$$

where  $\alpha, \beta^r, g_{rr}$  and  $g_{\theta\theta}$  are functions of  $r$  and  $t$ , and where  $\beta_r = g_{rr} \beta^r$ . Equations (12) and (13) now have only two independent components each. The  $rr$  and  $\theta\theta$  components of equation (12) are respectively

$$\frac{1}{2} \beta^r \frac{dg_{rr}}{dr} + g_{rr} \frac{d\beta^r}{dr} - \alpha K_{rr} - \sigma g_{rr} = 0, \quad (28)$$

$$\frac{1}{2} \beta^r \frac{dg_{\theta\theta}}{dr} - \alpha K_{\theta\theta} - \sigma g_{\theta\theta} = 0. \quad (29)$$

The  $rr$  and  $\theta\theta$  components of equation (13) are respectively

$$\begin{aligned} & -\frac{d^2 \alpha}{dr^2} + \frac{1}{2g_{rr}} \frac{dg_{rr}}{dr} \frac{d\alpha}{dr} + \beta^r \frac{dK_{rr}}{dr} + 2K_{rr} \frac{d\beta^r}{dr} \\ & + \alpha \left[ {}^{(3)}R_{rr} + K_{rr} \left( 2\frac{K_{\theta\theta}}{g_{\theta\theta}} - \frac{K_{rr}}{g_{rr}} \right) - \frac{\tau_{rr}}{2} + \frac{g_{rr}\tau_{\theta\theta}}{g_{\theta\theta}} - \frac{\rho g_{rr}}{2} \right] - \sigma K_{rr} = 0, \end{aligned} \quad (30)$$

$$-\frac{1}{2g_{rr}} \frac{dg_{\theta\theta}}{dr} \frac{d\alpha}{dr} + \beta^r \frac{dK_{\theta\theta}}{dr} + \alpha \left[ {}^{(3)}R_{\theta\theta} + \frac{K_{rr}K_{\theta\theta}}{g_{rr}} + \frac{1}{2} \left( \frac{\tau_{rr}}{g_{rr}} - \rho \right) g_{\theta\theta} \right] - \sigma K_{\theta\theta} = 0. \quad (31)$$

The constraint equations

$$^{(3)}R + K^2 - K_{ab}K^{ab} - 2\rho = 0, \quad D_a K^a_b - D_b K - J_b = 0, \quad (32)$$

become in spherical symmetry

$$\rho = \frac{{}^{(3)}R_{rr}}{2g_{rr}} + \frac{{}^{(3)}R_{\theta\theta}}{g_{\theta\theta}} + \frac{K_{\theta\theta}}{g_{\theta\theta}} \left( \frac{2K_{rr}}{g_{rr}} + \frac{K_{\theta\theta}}{g_{\theta\theta}} \right), \quad (33)$$

$$2 \frac{dK_{\theta\theta}}{dr} = -g_{\theta\theta} J_r + \left( \frac{K_{rr}}{g_{rr}} + \frac{K_{\theta\theta}}{g_{\theta\theta}} \right) \frac{dg_{\theta\theta}}{dr}. \quad (34)$$

### B. Generic coordinate conditions applied to spherical symmetry

In spherical symmetry, the conformal strain lapse, equation (23) is given by

$$\alpha = \frac{\frac{K_{rr}}{(g_{rr})^2} \left( \frac{1}{2} \beta^r \frac{dg_{rr}}{dr} + g_{rr} \frac{d\beta^r}{dr} \right) + \frac{K_{\theta\theta}}{(g_{\theta\theta})^2} \beta^r \frac{dg_{\theta\theta}}{dr} - \sigma \left( \frac{K_{rr}}{g_{rr}} + 2 \frac{K_{\theta\theta}}{g_{\theta\theta}} \right)}{\left( \frac{K_{rr}}{g_{rr}} \right)^2 + 2 \left( \frac{K_{\theta\theta}}{g_{\theta\theta}} \right)^2}. \quad (35)$$

The exponential curvature lapse, equation (25), is given by

$$\begin{aligned} & -\frac{d^2\alpha}{dr^2} + \left( \frac{1}{2g_{rr}} \frac{dg_{rr}}{dr} - \frac{1}{g_{\theta\theta}} \frac{dg_{\theta\theta}}{dr} \right) \frac{d\alpha}{dr} \\ & + \alpha g_{rr} \left[ \left( \frac{K_{rr}}{g_{rr}} \right)^2 + 2 \left( \frac{K_{\theta\theta}}{g_{\theta\theta}} \right)^2 + \frac{1}{2} \left( \rho + \frac{\tau_{rr}}{g_{rr}} + 2 \frac{\tau_{\theta\theta}}{g_{\theta\theta}} \right) \right] \\ & + g_{rr} \left[ \beta^r \frac{d}{dr} \left( \frac{K_{rr}}{g_{rr}} + 2 \frac{K_{\theta\theta}}{g_{\theta\theta}} \right) + \sigma \left( \frac{K_{rr}}{g_{rr}} + 2 \frac{K_{\theta\theta}}{g_{\theta\theta}} \right) \right] = 0. \end{aligned} \quad (36)$$

The minimal strain shift, equation (17), is given by

$$\begin{aligned} & \frac{d^2\beta^r}{dr^2} + \frac{d\beta^r}{dr} \left( \frac{1}{2g_{rr}} \frac{dg_{rr}}{dr} + \frac{1}{g_{\theta\theta}} \frac{dg_{\theta\theta}}{dr} \right) \\ & + \frac{\beta^r}{2} \left[ \frac{1}{g_{rr}} \frac{d^2g_{rr}}{dr^2} + \left( \frac{1}{g_{\theta\theta}} \frac{dg_{\theta\theta}}{dr} - \frac{1}{g_{rr}} \frac{dg_{rr}}{dr} \right) \frac{1}{g_{rr}} \frac{dg_{rr}}{dr} - \left( \frac{1}{g_{\theta\theta}} \frac{dg_{\theta\theta}}{dr} \right)^2 \right] \\ & - \left[ \frac{d}{dr} \left( \frac{\alpha K_{rr}}{g_{rr}} \right) + \frac{\alpha}{g_{\theta\theta}} \frac{dg_{\theta\theta}}{dr} \left( \frac{K_{rr}}{g_{rr}} - \frac{K_{\theta\theta}}{g_{\theta\theta}} \right) \right] = 0. \end{aligned} \quad (37)$$

Thus, in spherical symmetry the generalized SY gauge is the combination of equations (36) and (37), while the generalized BCT gauge is the combination of equations (35) and (37).

### C. Coordinate conditions tailored to spherical symmetry

In spherical symmetry, one can also find simpler equations for the lapse and the shift. We note that equation (29) contains no derivatives of the shift, while equation (31) contains only one derivative of the lapse. We can use equation (29) to express the shift as

$$\beta^r = 2 \frac{\sigma g_{\theta\theta} + \alpha K_{\theta\theta}}{dg_{\theta\theta}/dr}. \quad (38)$$

For  $\sigma = 0$  this is called the area freezing shift, as it keeps the area of a surface for constant  $r$  constant. Similarly, we call (31) the area lapse. Substituting (38) into (31), we obtain a first-order differential equation for  $\alpha$  alone. We can use the momentum and Hamiltonian constraints (34, 33) to simplify it to the form

$$\frac{d\alpha}{dr} = \frac{g_{\theta\theta}}{dg_{\theta\theta}/dr} \left[ \alpha \left( -{}^{(3)}R_{rr} - \frac{2g_{rr}K_{\theta\theta}J_r}{dg_{\theta\theta}/dr} + \tau_{rr} + \rho g_{rr} \right) + 2\sigma \left( K_{rr} - \frac{g_{rr}g_{\theta\theta}J_r}{dg_{\theta\theta}/dr} \right) \right]. \quad (39)$$

We can integrate this first-order ordinary differential equation from the center of spherical symmetry outwards. However, the equation requires as a boundary condition the value of the lapse at one value of  $r$ . We refer to the combination of equations (38) and (39) as the area gauge.

## D. Boundary conditions for the lapse and shift equations in spherical symmetry

In three space-dimensions without spherical symmetry, some of the equations for the lapse and shift we have discussed are coupled elliptic equations. These require boundary values for the lapse and shift on a surface (topologically a 2-sphere) at large radius. One would expect the equivalent problem in spherical symmetry to be a second-order ODE boundary value problem, with one boundary condition each at  $r = 0$  and large  $r$ , for each of  $\alpha$  and  $\beta^r$ . One boundary condition at  $r = 0$  is immediately obtained from symmetry, namely that  $\alpha$  must have an expansion in even powers of  $r$  and  $\beta^r$  an expansion in odd powers of  $r$ . For the second boundary condition, it is as natural to specify  $\alpha$  and  $\lim(\beta^r/r)$  at  $r = 0$  as it would be to specify  $\alpha$  and  $\beta^r$  at some large value of  $r$ . Doing that, we have the great simplification of integrating an ODE system from  $r = 0$  outwards, instead of solving a boundary value problem.

The equations for  $\alpha$  and  $\beta^r$  are homogeneously linear in  $\alpha$ ,  $\beta^r$  and  $\sigma$ . The boundary values we need can be thought of as an overall factor in both  $\alpha$  and  $\beta^r/r$ , and their ratio. The ratio is uniquely determined at the center of spherical symmetry. In general we have

$$-\alpha K + D_a \beta^a = \frac{\partial(\ln \sqrt{g})}{\partial t} = 3\sigma, \quad (40)$$

where the first equality follows from the definition of  $K$ , while the second equality follows from (12). At the center of spherical symmetry, this yields

$$\beta^r = \left( \sigma + \frac{1}{3} K \alpha \right) r + O(r^3). \quad (41)$$

The same result is also obtained by evaluating either (21), (23) or (38) at  $r = 0$ .

It remains to determine the overall factor in both  $\alpha$  and  $\beta^r$ . From (4) we have that  $\alpha = \alpha_0 = \text{const.}$  at the origin in the Killing case. The value of  $\alpha_0$  is a matter of convention that only affects the norm of the Killing vector field, and we can set it to any value that is constant in  $t$ .

In the self-similar case, we formally have  $\alpha = \alpha_0 \exp \sigma t$  at the center, but this result is not very useful, as only one choice of  $\alpha_0$  gives rise to a compatible coordinate system. To see this, label the central geodesic by proper time  $T$  with the origin chosen so that  $T = 0$  is the singular point of the self-similar spacetime. In the self-similar time coordinate  $t$  this point is  $t = \infty$ . Let  $T = T_0$  where the initial slice  $t = 0$  intersects the central geodesic. Then it is easy to see that the proper relation  $T = T_0 \exp \sigma t$  is obtained only if we choose  $\alpha_0 = \sigma T_0$ . In other words, we must guess correctly how far the slice is away from the singular point.

A prescription for  $\alpha$  at the center can be obtained by noting that we have  $\mathcal{L}_X T_{ab} = 0$  and therefore  $\mathcal{L}_X \rho = -2\sigma \rho$ . In the 3+1 split this is

$$(\alpha n^a + \beta^a) \nabla_a \rho = -2\sigma \rho. \quad (42)$$

We combine this with the equation for  $n^a \nabla_a \rho$  obtained from stress-energy conservation,

$$0 = -n_a \nabla_b G^{ab} = -K \rho + n^a \nabla_a \rho + D_a J^a - \tau^{ab} K_{ab} + 2J^a D_a \ln \alpha. \quad (43)$$

At the center of spherical symmetry,  $D_a \alpha$  and  $\beta^a$  both vanish, and we obtain an algebraic equation for  $\alpha$  alone. Furthermore at the center we have  $K_{ab} = (K/3)g_{ab}$  and a similar expression for  $\tau_{ab}$ , which simplifies our final result to

$$\alpha = \frac{2\sigma \rho g_{rr}}{3 \left[ \frac{dJ_r}{dr} - K_{rr} \left( \rho + \frac{\tau_{rr}}{g_{rr}} \right) \right]} + O(r^2) \quad (44)$$

This equation gives a (nonzero) value for  $\alpha$  at the origin provided that neither the numerator nor the denominator vanishes there.

## IV. TESTS IN SPHERICAL SYMMETRY

### A. Numerical method

We now discuss empirical tests of some of the coordinate conditions we have discussed for some given spacetimes, restricting ourselves to spherically symmetric examples. Rather than to evolve Cauchy data via the ADM equations,

we evolve an embedded slice in the spacetime. All indices  $A, B, C, i, j, k$  in this section denote coordinate components of tensors, rather than abstract tensor indices. Let a spacetime metric be given in closed form in coordinates  $x^A$ , giving rise to a metric  $g_{AB}$ . A three-dimensional spacelike slice embedded in the spacetime, together with a choice of three coordinates  $x^i$  on the slice, is parameterized by four functions of three coordinates,  $x^A = F^A(x^i)$ . The four-metric  $g_{AB}$  induces a three metric  $g_{ij}$  and extrinsic curvature  $K_{ij}$ , and there is a unit timelike normal  $n^A$  to the slice at each point.

In our code, we choose a spacetime, and an initial slice, and calculate  $g_{ij}$  and  $K_{ij}$ . The equations for doing this are derived in full in [7]. If the spacetime contains matter, the components  $\rho$ ,  $J_i$  and  $S_{ij}$  of the stress-energy tensor are also computed from  $T_{AB}$ . These are handed to a coordinate condition solver that returns  $\alpha$  and  $\beta^i$ . The slice is then evolved via

$$\frac{\partial F^A}{\partial t} = \alpha n^A + \beta^i \frac{\partial F^A}{\partial x^i}. \quad (45)$$

We finite-difference the equations using a type of iterative Crank-Nicholson (ICN) algorithm.  $\partial F^A/\partial t$  is computed at time  $t_n$ , and the  $F^A$  are evolved to  $t_{n+1}$  with a forward in time, centered in space (predictor) step.  $\partial F^A/\partial t$  is then computed at time  $t_{n+1}$ , and the average of  $\partial F^A/\partial t$  at  $t_n$  and  $t_{n+1}$  is then used for a corrector step. The corrector step is iterated to convergence. To maintain stability, we use numerical viscosity.

In spherical symmetry, there are only two functions  $X^0(r)$  and  $X^1(r)$ , and two components each of the induced metric and extrinsic curvature,  $g_{rr}(r)$ ,  $g_{\theta\theta}(r)$ ,  $K_{rr}(r)$ , and  $K_{\theta\theta}(r)$ , and stress-energy components  $\rho(r)$ ,  $J_r(r)$  and  $\tau_{rr}(r)$ . There is only one shift component,  $\beta^r(r)$ , and the lapse,  $\alpha(r)$ . The coordinate condition solver and slice evolution code are independent and communicate only through the fields intrinsic to the slice, so that the coordinate condition solver never sees the  $F^A$  directly. The code assumes that surfaces of constant  $t$  are spacelike, and complains if, through a bad choice of coordinate, they are not, but no assumptions are made on the coordinates  $X^0$  and  $X^1$ . They could, for example, be null coordinates.

Integrating the embedding of a slice, rather than Cauchy data, has two advantages. The first one is stability: empirically, the ICN algorithm, together with a linear extrapolation outer boundary condition, is very stable for given  $\alpha$  and  $\beta^r$ . In particular, if we want to excise the center of spherical symmetry from the numerical domain in a black hole spacetime, an extrapolation inner boundary works well in the slice evolution code, while black hole excision in a genuine spacetime evolution code is a nontrivial problem even in spherical symmetry. The second advantage of the slice evolution is that we can use a given spacetime that is not a solution of the Einstein equations for any reasonable matter, but that has some properties worth investigating. In such a spacetime, we create a fictitious stress-energy tensor simply defined by the Einstein equations but not derived from any reasonable matter source. We think that such “fake” spacetimes can be valuable tests of qualitative aspects of coordinate conditions.

## B. Stationary spacetimes

We have used a variety of spacetimes to test three different gauges: area gauge, generalized SY gauge and generalized BCT gauge. The results are as follows.

As model stationary spacetimes with a regular center we have examined Minkowski space, de Sitter space and the Schwarzschild solution for a constant density star, parameterized by  $R_{\text{star}}/M_{\text{star}} < 9/4$ . All three gauge choices work on the Minkowski and de Sitter spacetimes. The area and generalized SY gauges work on the constant density star. The generalized BCT gauge fails numerically at the boundary of the star, where  $dg_{rr}/dr$  is discontinuous. As a boundary condition we choose  $\alpha = 1$  at the center. As the equations are linear in both  $\alpha$  and  $\beta^r$ , this choice is equivalent to any other choice that is constant in time. As a consistency check of our programming, we have verified that the Hamiltonian and momentum constraints are obeyed up to finite differencing error. As the initial slice, we take  $X^1 = r$  and  $X^0(r)$  a Gaussian, with  $X^0$  and  $X^1$  the standard time and radial coordinates. We find that the induced metric, extrinsic curvature, stress-energy components, lapse and shift are all time-independent up to numerical error.

To test coordinate conditions in a black hole spacetime with the black hole excised from the numerical domain, we have implemented the Schwarzschild spacetime in coordinates that are regular at the future event horizon (the Painlevé-Gullstrand coordinates). Here the area gauge and generalized SY gauge work. The generalized BCT gauge has numerical problems at the excision boundary. As a boundary condition we have chosen  $\alpha = 1$  at the excision boundary, which is chosen at constant coordinate  $r$ . As the shift is supposed to keep our generic radial coordinate  $r$  at constant curvature radius  $\sqrt{g_{\theta\theta}}$ , this boundary condition is consistent. As the initial slice, we took  $X^1 = r$  and  $X^0(r)$  a Gaussian, with  $X^1$  the area radius and  $X^0$  the Painlevé-Gullstrand time (which is defined by surfaces of constant  $X^0$  being flat and spacelike). Again we find that all variables are constant in time, even though they are all nontrivial.



### C. Self-similar spacetimes

We have also investigated four self-similar spacetimes. Our coordinate prescriptions are designed to bring the metric of a CSS spacetime into the form

$$g_{\mu\nu} = e^{-2t} \bar{g}_{\mu\nu} \quad (46)$$

where the components of  $\bar{g}_{\mu\nu}$  are independent of  $t$ . The metric of a DSS spacetime can be written in the same form, with  $\bar{g}_{\mu\nu}$  periodic in  $t$ . It is important to note that while all our prescriptions must work on CSS spacetimes, barring numerical error and numerical instability, it is not guaranteed that they work, in the sense of bringing the metric into this form, on DSS spacetimes. The reason is that given a CSS spacetime and an initial slice, the coordinate system of this form is unique, while it is not for a DSS spacetime. In order to see how successful a gauge is in making the conformal spacetime metric independent of  $t$  or periodic in  $t$ , it is useful to rescale fields intrinsic to the slice ( $g_{ij}$ ,  $K_{ij}$  etc.) with appropriate powers of  $\exp t$  so that the rescaled fields are independent of time if and only if the spacetime is CSS and our algorithm finds the conformal Killing vector.

The simplest CSS spacetime, Minkowski spacetime, is not suitable for our investigation, as the homothetic vector is not unique. Instead, we have considered a conformally flat spacetime of the form

$$ds^2 = e^{-2t} (-dt^2 + dr^2 + r^2 d\Omega^2). \quad (47)$$

We have given this spacetime fictitious matter defined by the Einstein equations. It has a unique homothetic vector field if by assumption we restrict that vector field to the  $tr$  plane. Here, we have tested all three gauges. All work well when we use the analytic boundary condition  $\alpha = e^{-t}$  at the center.

A less trivial example is the Roberts [8] solution, a family of CSS massless scalar field solutions, which in double null coordinates is given by

$$ds^2 = -du dv + \frac{1}{4} [(1-p^2)v^2 - 2uv + u^2] d\Omega^2 \quad (48)$$

$$\phi(u, v) = \ln \frac{(1-p)v - u}{(1+p)v - u}, \quad (49)$$

with  $p$  a parameter. This spacetime has curvature singularities at  $g_{\theta\theta} = 0$ . We have investigated the case  $p = 0.5$ , where these singularities are naked, and lie in the future and past lightcones. The slice we evolve lies in the spacelike sector. The only boundary condition for  $\alpha$  at the inner boundary that worked here is  $\alpha = \alpha_0 \exp \sigma t$ , with  $\alpha_0$  taken from the exact solution. Here the area gauge and generalized SY gauge work. The generalized BCT gauge has numerical problems at the excision boundary.

Our other two examples are of more physical interest. They are the CSS critical solution for the collapse of a perfect fluid with the equation of state  $p = (1/3)\rho$  [9] and the DSS critical solution for the collapse of a massless scalar field [10]. These solutions have been constructed numerically as solutions of a nonlinear ODE eigenvalue problem [9], and a nonlinear PDE eigenvalue problem [11], respectively. We interpolate them numerically to the required values of  $X^0$  and  $X^1$ . For both spacetimes, we have tested the area gauge and the generalized SY gauge. Again, we use a Gaussian initial slice.

In the CSS fluid spacetime, we have been able to use the boundary condition (44). The boundary condition works, and all variables are constant in time.

The DSS scalar field spacetime has a regular center, but the scalar field is a periodic function of  $\tau$  at the center, so that the density  $\rho$  vanishes twice per period. This means that we cannot apply the boundary condition (44). Instead, we have applied the exact boundary condition  $\alpha = \alpha_0 \exp \sigma t$ , with  $\alpha_0$  taken from the exact solution. This test is nontrivial, as it is not clear at all that a coordinate condition designed to catch a continuous symmetry can catch a discrete one. But it works: all variables are periodic in time for the scalar field DSS spacetime for at least two periods, in both the area and generalized SY gauge.

## V. CONCLUSIONS

We have examined the problem of finding good coordinates in numerical relativity, and the problem of defining a renormalisation group for general relativity in critical collapse, in a single framework. We have listed old and new prescriptions that obey our criterion for a good coordinate condition in either application: that the presence of an approximate timelike Killing field, or an approximate homothetic vector field, in some region of the spacetime (for example at late times) be revealed by metric coefficients that vary only slowly with time.

We have tested several prescriptions numerically in spherically symmetric spacetimes that are exactly stationary or homothetic. For these exact symmetries, it is clear analytically that our prescriptions will work, but we have used these cases both to test our code, and to see how well different schemes can be implemented numerically. We find that a scheme specific to spherical symmetry (area freezing shift and radial lapse) is the most stable one, generalized Smarr-York (SY) gauge (minimal strain shift and exponential  $K$  lapse) comes second, while we have been unable to implement the generalized Brady-Crighton-Thorne (BCT) gauge (minimal strain shift and conformal strain lapse) stably. What this means is that the combination of our numerical methods of solving for the BCT gauge and of evolving a slice is numerically unstable. This indicates that coupling the BCT gauge to a genuine spacetime evolution code may also be numerically unstable, but it is of course possible that a different numerical implementation of the gauge and/or the evolution equations is stable. Other difficulties of the BCT gauge are evident analytically: Not all initial data sets (in our case, initial slices through a given spacetime) admit a solution to BCT gauge, and BCT gauge may still break down (when  $K_{ab}K^{ab} = 0$  anywhere) at a later time. In spherical symmetry, the slice must nowhere be close to a polar slice for BCT gauge to work.

We have entered new terrain examining one spacetime (the Choptuik scalar field critical solution) with a discrete, rather than continuous, symmetry. Here we found numerically that two prescriptions made for a continuous symmetry (area gauge and generalized SY gauge) actually follow the discrete symmetry, in the sense that the metric becomes periodic in time. (Our third gauge, generalized BCT gauge, was unstable numerically in our implementation, but may yet work with better numerical methods.) This is an important step in the construction of a renormalisation group for critical collapse, as it suggests that a single coordinate prescription may realize the renormalisation group both for continuously and discretely self-similar (CSS and DSS) critical solutions. It is also encouraging for the use of co-rotating coordinates in non-circular binary black hole inspiral.

One crucial ingredient is still lacking in a working prescription for a renormalisation group, namely the boundary conditions used when solving elliptic equations for the lapse and shift. Details aside, these boundary conditions must contain a good guess for how far away in proper time the slice is from the accumulation point of the (approximately) self-similar spacetime. Although there are ideas [see equation (44)], we have not been able to solve this problem, and instead had to supply this information by hand.

Future work should include testing beyond spherical symmetry and in particular with angular momentum, and on spacetimes with approximate symmetries, as well as developing ideas on what boundary conditions should be imposed on elliptic coordinate conditions.

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